Technical Notes

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Free Jet Expansion in a Rapidly Rotating Gas

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I. Introduction

IN the gas centrifuge for enrichment of uranium, the feed gas is introduced into the central part of the rapidly rotating cylinder and expands into a rotating ambient gas at extremely low pressure in the form of a free jet. The calculation of the shape of the jet is required to provide a boundary condition for determining the velocity distribution in the rotating cylinder.

In the present Note, the Newtonian thin shock layer approximation[1-6] is applied to calculate the inviscid structure of the free jet exhausting into a rotating gas. The use of a Newtonian model is justified, since exact calculation of jet flowfield is unnecessary and the value of specific heat ratio of uraniumhexafluoride gas is close to unity.

II. Equations for Jet Boundary and Barrel Shock Shape

A solution to the equations of motion in the shock layer is given by Chernyi,[7] using an asymptotic expansion in terms of power series in the small parameter, ϵ, where $\epsilon=(\gamma-1)/(\gamma+1)$. The first approximate solutions to the flow between jet boundary and barrel shock become

$$u=u(\psi) \tag{1}$$

$$P^{1/\gamma}=c(\psi)\rho \tag{2}$$

$$P=\frac{1}{R_c R_T}\int_{\psi_S}^{\psi} u\mathrm{d}\psi+P_S(x) \tag{3}$$

$$y=\frac{1}{R_T}\int_0^{\psi}\frac{\mathrm{d}\psi}{\rho u} \tag{4}$$

$$v=u\frac{\partial y}{\partial x} \tag{5}$$

where x and y are the coordinates in directions parallel and normal, respectively, to the jet boundary with the origin at the edge of nozzle exit; ψ is the stream function; u and v are the velocities in the x and y directions; the longitudinal and transverse radius of curvature are denoted by R_c and R_T, respectively; and s represents the quantities on the immediate downstream side of the barrel shock.

The equations for jet boundary and barrel shock shape can be derived from Eqs. (1-5) if an analytical representation of the undisturbed flow inside the jet is given. As has been shown by several authors,[3,8] a free jet is similar to a source flow, and Boynton's formula[3] for the angular dependance of density is used to infer the flow variables in the jet. When the fluid velocity does not acquire its limiting value, the formula can be modifed in the form as

$$\frac{\rho q}{\rho_{\mathfrak{t}} q_{\mathfrak{t}}}=\cos^{2/\gamma-1}\left(\frac{\pi}{2}\frac{\theta}{\theta_m}\right) \tag{6}$$

where θ denotes the polar angle measured from the axis of jet, θ_m is the maximum Prandtl-Meyer expansion angle, and $\rho_{\mathfrak{t}}$ and $q_{\mathfrak{t}}$ denote the density and the velocity on the axis, respectively. In the neighborhood of the jet axis, the analogy to a simple source flow is strong[8]; hence, $\rho_{\mathfrak{t}}$ and $q_{\mathfrak{t}}$ can be determined by using the following implicit expression for Mach number,

$$\frac{\left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)}M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{(\gamma+1)/2(\gamma-1)}}=\frac{1}{X^2} \tag{7}$$

where X denotes the dimensionless axial distance from the apparent source, which is nondimensionalized by the sonic radius r^*. In the following, all variables will be made dimensionless by the relevant quantites under the sonic condition and the sonic radius r^*.

From Eq. (6), we obtain

$$\mathrm{d}\psi=\left[\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_m}\right)\right]^{2/(\gamma-1)}\sin\theta\mathrm{d}\theta \tag{8}$$

Substitution of Eq. (8) into Eq. (3) yields

$$P_\infty=\frac{1}{R_c r\sin\theta}\int_\theta^{\theta_m}u\left[\cos\left(\frac{\pi}{2}\frac{\xi}{\theta_m}\right)\right]^{2/(\gamma-1)}\sin\xi\mathrm{d}\xi+P_s \tag{9}$$

where P_∞ is the pressure on jet boundary and r, θ are the polar coordinates of jet boundary. The gas in the rotating cylinder is assumed to be isothermal and of rigid body rotation with angular velocity ω, which leads to the pressure on the boundary of the form:

$$P_\infty=P_a\exp\{\tfrac{1}{2}M_R^2[(\beta r\sin\theta)^2-1]\} \tag{10}$$

where P_a is the peripheral pressure in the rotating cylinder; $\beta=r^*/a$, the ratio of the sonic radius to the radius of cylinder (r has been made dimensionless by r^*) and $M_R=a\omega/\sqrt{(1/M)RT}$ is the peripheral almost Mach number.

At some station (r_0,θ_0) near th exit, the shock layer is assumed homogeneous, and the fluid velocity and density are taken to be the local values behind the barrel shock. Thus, the integral appearing in Eq. (9) is split in the form as

$$P_\infty=\frac{1}{R_c r\sin\theta}[q_0\cos(\theta_0-\phi_0)\int_{\theta_0}^{\theta_m}\left[\cos\left(\frac{\pi}{2}\frac{\xi}{\theta_m}\right)\right]^{2/(\gamma-1)}\times\sin\xi\mathrm{d}\xi+I]+P_s \tag{11}$$

$$I=\int_\theta^{\theta_0}q\cos(\xi-\phi)\left[\cos\left(\frac{\pi}{2}\frac{\xi}{\theta_m}\right)\right]^{2/(\gamma-1)}\sin\xi\mathrm{d}\xi \tag{12}$$

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where the inclination of jet boundary is denoted by ϕ. Equation (11) in conjunction with the Newtonian formula for the pressure behind oblique shock wave gives the explicit expression for R_c. The result is

$$R_c = \frac{r}{\sin\theta} \times \frac{q_0\cos(\theta_0-\phi_0)\int_{\theta_0}^{\theta_m}\left[\cos\left(\frac{\pi}{2}\frac{\xi}{\theta_m}\right)\right]^{2/(\gamma-1)}\sin\xi d\xi + I}{(P_\infty - P)r^2/\gamma - q\cos(\theta-\phi)\sin^2(\theta-\phi)\left[\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_m}\right)\right]^{2/(\gamma-1)}} \tag{13}$$

where P and q denote, respectively, the undisturbed pressure and the undisturbed velocity immediate upstream side of the barrel shock. Equation (13) is equivalent to Hubbard's Eq. (3),[2] whereas the pressure P_∞ varies along the jet boundary, according to Eq. (10) due to the rotation of ambient gas. The undisturbed velocity q is also a variable in the present formulation. Two additional geometrical equations are available by virtue of the thin shock layer assumption;

$$\frac{d\theta}{dr} = \frac{1}{r}\tan(\phi-\theta) \tag{14}$$

$$\frac{d\phi}{dr} = -\frac{1}{R_c}\frac{1}{\cos(\phi-\theta)} \tag{15}$$

Writing Eq. (12) in the form of differential equation

$$\frac{dI}{dr} = \frac{q}{r}\cos^{2/(\gamma-1)}\left(\frac{\pi}{2}\frac{\theta}{\theta_m}\right)\sin\theta\sin(\theta-\phi) \tag{16}$$

we obtain a set of differential equations, (14-16) to be solved for $\theta(r)$, $\phi(r)$ and $I(r)$. The jet boundary shape is determined from the functional form of $\theta(r)$.

The thickness of the shock layer δ was estimated by using Eq. (4) in the following manner. From Eq. (2), we get

$$\rho = \rho_s(\xi)[P/P_s(\xi)]^{1/\gamma} \tag{17}$$

where $\rho_s(\xi)$ and $P_s(\xi)$ represent the density and pressure behind the shock at the station $[r(\xi),\xi]$ at which the relevant streamline entered the shock layer. On the one hand, Eqs. (3) and (8) provide the expression for the pressure distribution:

$$P = P_s(r,\theta) + \frac{\gamma}{R_c r\sin\theta}\int_\xi^\theta q\cos(\xi-\phi)\left[\cos\left(\frac{\pi}{2}\frac{\xi}{\theta_m}\right)\right]^{2/(\gamma-1)}\sin\xi d\xi \tag{18}$$

where each streamline is associated with the polar angle ξ at its entry point. Thus, with the aid of the relation for the density ratio across the shock, and using Eq. (4), we obtain an expression for the shock layer thickness, δ:

$$\delta = \frac{1}{r\sin\theta}\left\{\int_\theta^{\theta_0}\left[\frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M^2\sin^2(\xi-\phi)}\right] \times \frac{r^2\sin\xi}{\cos(\xi-\phi)(P/P_s)^{1/\gamma}}d\xi + q_0\cos(\theta_0-\phi_0) \times \int_{\theta_0}^{\theta_m}\frac{1}{\rho_{s0}(P/P_{s0})^{1/\gamma}}\cos^{2/(\gamma-1)}\left(\frac{\pi}{2}\frac{\xi}{\theta_m}\right)\sin\xi d\xi\right\} \tag{19}$$

If the solution to Eqs. (14-16) is given, we can calculate δ from Eq. (19) by numerical quadrature.

III. Numerical Results and Discussions

The jet boundary shapes were determined by the numerical integration of Eqs. (14-16) with the aid of Eq. (6), using the Runge-Kutta-Gill method. Then, barrel shock shapes were calcuated by employing Eq. (19) with the aid of jet boundary geometry. The initial station for integration of the basic equations is taken at the point near the exit where the radius of curvature of the boundary given by Eq. (13) coincides with the initial radius of curvature of the jet R_{co}, on the assumption that the initial radius of curvature of jet is maintained a small distance downstream of nozzle exit. Johannesen and Meyer[9] derived the formula for the initial curvature of the jet boundary using their solution to the axially-symmetric supersonic flow near the center of an expansion. This formula can be applied to the jet issuing into a rotating gas, and it becomes

$$\frac{1}{R_{co}} = -[q_F\sin\theta_j\sin^2\mu_F + u_I(\phi_F)]/(q_F\sin 2\mu_F) + \frac{1}{q_F}\cot\mu_F\left(\frac{dq}{dr}\right)_{R=0} \tag{20}$$

$$\left(\frac{dq}{dr}\right)_{R=0} = -\sin\theta_j\sqrt{\frac{\gamma+1}{2}}\frac{1}{\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{3/2}} \times \frac{1}{\gamma\left(\frac{\gamma+1}{2}\right)M_\infty\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{1-2\gamma/\gamma-1}}\frac{P_\infty}{P^*}M_R^2\left(\frac{D/2}{a}\right)^2 \tag{21}$$

where M_∞ denotes Mach number on the boundary, and the nomenclature for other variables used in Eq. (20) is defined in Ref. 9.

Figure 1 shows the plot of the initial radius of curvature against the expansion pressure ratio with the rotational almost Mach number and the ratio of the nozzle radius to the rotor radius as parameters. It is seen that rotational effect decreases the value of the radius of curvature.

In Figs. 2 and 3 the boundary and shock wave shapes for the jet exhausting into a rotating gas are plotted and compared with the corresponding results obtained by the method of characteristics.[10]† To compare the results of present computation with those from characteristics calculation, the following relation is employed:

$$\frac{r^*}{D/2} = 1\Big/\sqrt{2\int_0^{\theta_m}\left[\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_m}\right)\right]^{2/\gamma-1}\sin\theta d\theta} \tag{22}$$

Equation (22) is obtained by equating the mass flux through the nozzle to the total mass flux of the modified source flow.[11] For $\gamma = 5/3$, Eq. (22) yields $r^*/(D/2) = 1.36$, while Muntz[12] gave the same value by matching the temperature of simple source flow expansion to the jet centerline temperature obtained from characteristic calculation. For $\gamma = 1.06$, $r^*/(D/2)$ becomes 0.948. We see, from Figs. 2 and 3, that the approximate technique based on the Newtonian thin layer assumption gives fairly satisfactory results for the boundary and barrel shock wave location. For large expansion pressure ratio, there exists a discrepancy in the region far downstream. This error may be corrected by successive approximation as has been demonstrated by Mitome and Yasuhara,[6] in the case of jet plume exhausting into the atmosphere.

†The programming details may be obtained by direct correspondence with Mr. Hasegawa, c/o Hitachi Works of Hitachi LTD. 3-1-1 Saiwai-cho, Hitachi-Shi, Ibaraki-ken, 317, Japan.

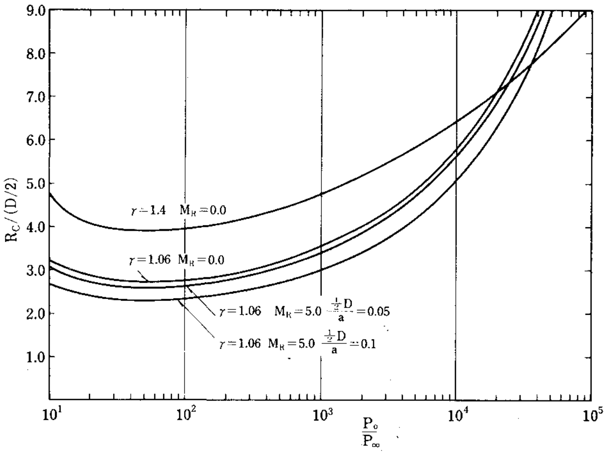


Fig. 1 Effects of expansion pressure ratio and rotational Mach number upon the initial radius of curvature of jet boundary.

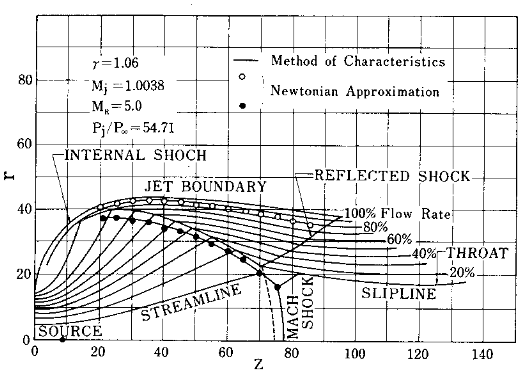
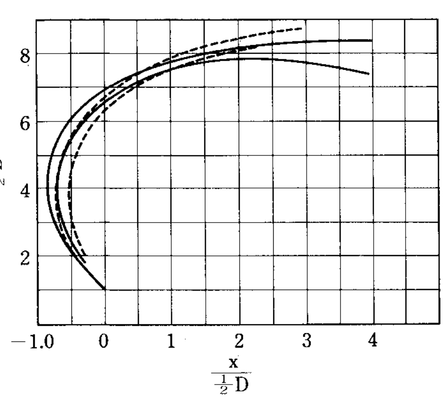


Fig. 2 Comparison of the Newtonian shock layer solution with the numerical solution by the method of characteristics. The location of Mach shock is determined by the Eastman-Radtke scheme[13] in the Newtonian approximate calculation.



$P_j/P_\infty = 632.3$, $\frac{1}{2}D/a = 0.05$

$M_R = 5$, $x_o/\frac{1}{2}D = 1.095$

Fig. 3 Comparison of the Newtonian shock layer solution with the numerical solution by the method of characteristics. ——Newtonian approximation,--- method of characteristics.

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Static Stability Analysis of Elastically Restrained Structures under Follower Forces

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I. Introduction

IN this Note, elastic undamped systems carrying no attached mass and subjected to follower forces of constant magnitude are considered. These systems may be of the divergence or flutter type depending on the boundary conditions.[1] Necessary conditions for flutter and divergence instability of elastic systems under follower forces are derived in Ref. 2. Classical examples of nonconservative (in the broader sense) structural systems which can be treated as divergence type structures are Pflüger's column and Greenhill's beam.[3] In the last reference, a lower bound theorem for divergence type systems is presented using the dynamic method. According to this theorem, the critical load of a divergence type system under a follower force for certain boundary conditions is greater than the critical load of the corresponding conservative system (which is subjected to a constant directional force).

In this investigation, using as models elastically restrained simple structures under follower compressive forces, it is shown that their type of instability is dependent on the values of the constants of elastic restraint. For some values of these

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